

# Directed graphs and Lie superalgebras of matrices

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# LIE SUPERALGEBRAS OF MATRICES

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# 1 Introduction

Let  $F$  be a field,  $\mathbb{F}_2 = \{0, 1\}$  the field of order two, and let  $\mathbb{Z}_2$  be the additive group of  $\mathbb{F}_2$ . An algebra  $A$  over  $F$  is called a *superalgebra* if it is the direct sum of  $F$ -spaces  $A_0$  and  $A_1$  such that  $A_a A_b \subseteq A_{a+b}$  for all  $a, b \in \mathbb{Z}_2$ . The concept of a superalgebra has played crucial roles in solutions to a number of difficult problems. Serious attention in the literature has been devoted to superalgebras also in view of valuable applications related to physics. Without trying to give a survey we refer to a few recent papers ([2], [11], [15], and [16]) dealing with superalgebras.

A superalgebra  $L = L_0 + L_1$  is called a *Lie superalgebra* if the following conditions are satisfied, for all  $a, b \in \{0, 1\}$ ,  $x \in L_a$ ,  $y \in L_b$ ,  $z \in L$ ,

$$[x, y] = -(-1)^{ab}[y, x], \quad (1)$$

$$[[x, y], z] = [x, [y, z]] - (-1)^{ab}[y, [x, z]]. \quad (2)$$

For a systematic exposition of earlier results on Lie superalgebras the reader is referred to [1] and [18] (see also [12], §9.3).

The aim of this paper is to describe Lie superalgebras represented by blocked matrices of directed graphs. The description is related to the more general problem of characterizing gradings of matrix algebras recorded in [12], Problem 10.2, and considered by several mathematicians (see, in particular, [3], [4], [6], [7]). Professor Efim Zelmanov mentioned this general problem to the author when the latter was a student (see [12], Chapter 10). The results of this paper were discussed during a seminar talk at the University of Wisconsin, when the author worked there on sabbatical in 2000. Professors Don Passman and Jim Osterburg suggested several substantial simplifications to the proofs. Earlier, the author had also learnt a lot from collaboration and discussions with Professor Sorin Dăscălescu visiting the University of Stellenbosch, and then the University of Tasmania. The present work had originally relied on the general techniques presented in [6].

## 2 Main Theorems

Throughout the word *graph* means a finite directed graph without multiple edges but possibly with loops, and  $D = (V, E)$  stands for a graph with the set  $V = \{1, 2, \dots, n\}$  of vertices and the set  $E$  of edges. Our main definition uses structural matrix algebras or blocked matrix algebras defined by graphs (see, for example, [5], [8], [9], [10], [13], [14], and [12], §3.14), where the edges of  $D$  correspond to the standard matrix units of the algebra  $M_n(F)$  of all  $(n \times n)$ -matrices over  $F$ . Namely, for  $(i, j) \in E \subseteq V \times V$ , let  $e_{(i,j)} = e_{i,j} = e_{ij}$  be the standard matrix unit with the only nonzero entry 1 in the  $i$ -th row and  $j$ -th column.

Let  $\alpha$  be a mapping from  $E$  to  $\mathbb{F}_2 = \{0, 1\}$ . For each  $i \in \{0, 1\}$ , put

$$E_i = E_i(D, \alpha) = \{w \in E \mid \alpha(w) = i\}, \quad (3)$$

$$L_i = L_i(D, \alpha) = \bigoplus_{w \in E_i} F e_w. \quad (4)$$

Then  $E = E_0 \dot{\cup} E_1$  is a disjoint union and  $L_0 \oplus L_1$  is a direct sum.

A Lie superalgebra  $L$  is called a *blocked matrix Lie superalgebra* if there exists a graph  $D = (V, E)$  and a mapping  $\alpha : E \rightarrow \mathbb{Z}_2$  such that  $L$  is isomorphic to the set

$$L = L(D) = L(D, \alpha) = L_0 \oplus L_1 = \bigoplus_{w \in E} F e_w \quad (5)$$

endowed with a super commutator defined, for all  $a, b \in \mathbb{F}_2$ ,  $x \in L_a$ ,  $y \in L_b$ , by the rule

$$[x, y] = xy - (-1)^{ab}yx. \quad (6)$$

Thus  $L(D, \alpha)$  consists of all matrices with nonzero entries corresponding to the edges of the graph  $D$ , and zeros in all entries for which there are no edges in  $D$ . The elements  $a$  and  $b$  are called the *parities* of  $x$  and  $y$ , respectively. Every element  $r$  of  $L(D, \alpha)$  has a unique representation as  $r = \sum_{w \in E} r_w e_w$ , where  $r_w \in F$ . The elements  $r_w e_w$  are called the *homogeneous* components of  $r$ .

Since this is a natural generalization of the standard way of introducing Lie superalgebras on the set of matrices (see [1], §1.6 and [12], §3.14, §9.3), an interesting question that arises is: Find necessary and sufficient conditions on  $D$  and  $\alpha$  for  $L(D, \alpha)$  to be a Lie superalgebra. We are going to answer this question by giving a complete description of all mappings  $\alpha$  such that  $L(D, \alpha)$  is a Lie superalgebra.

Let us begin with general conditions on  $D$  and  $\alpha$  characterizing blocked matrix Lie superalgebras  $L(D, \alpha)$ . Our first main theorem summarizes these conditions and shows that not all graphs are worth considering in this context, as some of them do not give any blocked matrix Lie superalgebras.

We say that the set  $E$  of edges of  $D = (V, E)$  is *transitive* if

$$(i, j), (j, k) \in E \Rightarrow (i, k) \in E, \quad (7)$$

for all  $(i, j), (j, k) \in E$ . Let  $G$  be an Abelian group in additive notation. A mapping  $\gamma : E \rightarrow G$  will be called a *homomorphism* if the identity

$$\gamma((i, k)) = \gamma((i, j)) + \gamma((j, k)) \quad (8)$$

is satisfied for all  $(i, j), (j, k), (i, k) \in E$ .

**Theorem 1** *Let  $D = (V, E)$  be a graph, and let  $\alpha : E \rightarrow \mathbb{Z}_2$  be a mapping. Then the following conditions are equivalent*

- (i) *the set  $L(D, \alpha)$  is a Lie superalgebra;*
- (ii)  *$L(D, \alpha)$  is an associative superalgebra with respect to matrix product;*
- (iii)  *$E$  is transitive and  $\alpha$  is a homomorphism.*

The following technical concept is crucial in describing all Lie superalgebras  $L(D, \alpha)$ .

**Definition 1** Let  $D = (V, E)$  be a graph. A subset  $B$  of  $E$  is called a *superbasis* of  $D$  if and only if every mapping  $\beta : B \rightarrow \mathbb{Z}_2$  uniquely extends to a mapping  $\alpha : E \rightarrow \mathbb{Z}_2$  such that  $L(D, \alpha)$  is a Lie superalgebra.

Each superbasis  $B$  of  $D$  immediately gives us a complete description of all blocked matrix Lie superalgebras  $L(D, \alpha)$ . Indeed, there is a one-to-one correspondence between all mappings  $\beta : B \rightarrow \mathbb{Z}_2$  and all blocked matrix Lie superalgebras  $L(D, \alpha)$ . For each mapping  $\beta : B \rightarrow \mathbb{Z}_2$ , our technical Propositions 1 and 2 in Section 3 show how to find the unique homomorphism  $\alpha : E \rightarrow \mathbb{Z}_2$  extending  $\beta$ . Therefore (3) and (4) define the Lie superalgebra  $L(D, \alpha)$ . In particular, for a graph  $D = (V, E)$  with superbasis  $B$  it follows that there exist  $2^{|B|}$  superalgebras  $L(D, \alpha)$ . The next corollary follows from Theorem 1.

**Corollary 1** *Let  $D = (V, E)$  be a graph with a transitive set of edges, and let  $B \subseteq E$ . Then  $B$  is a superbasis of  $D$  if and only if every mapping  $\beta : B \rightarrow \mathbb{Z}_2$  uniquely extends to a homomorphism  $\alpha : E \rightarrow \mathbb{Z}_2$ .*

Theorem 1 tells us that only graphs with transitive sets of edges may have superbases. The second main theorem of this paper establishes that each graph  $D = (V, E)$  with a transitive set of edges has a superbasis. Thereby it describes all blocked matrix Lie superalgebras  $L(D, \alpha)$ .

**Theorem 2** *Every finite directed graph with transitive set of edges possesses a superbasis.*

Our proof is constructive and gives an explicit algorithm for finding a superbasis. Examples 1 and 2 show that graphs may have several superbases.

### 3 Technical Propositions and Proofs

*Proof of Theorem 1.* (i) $\Rightarrow$ (iii): Suppose that  $L(D, \alpha)$  is a Lie superalgebra. Take any

$i, j, k \in V$  with  $(i, j), (j, k) \in E$ .

Clearly,  $e_{i,j}$  and  $e_{j,k}$  belong to  $L(D, \alpha)$  by definition. Hence the commutator  $[e_{i,j}, e_{j,k}]$  lies in  $L(D, \alpha)$  too. Let  $a = \alpha(e_{i,j})$  and  $b = \alpha(e_{j,k})$ . Then

$$[e_{i,j}, e_{j,k}] = e_{i,j}e_{j,k} - (-1)^{ab}e_{j,k}e_{i,j} = e_{i,k} - (-1)^{ab}e_{j,k}e_{i,j}.$$

If  $(i, k) = (j, j)$ , then  $i = j = k$  and the implications (7) and (8) are trivial. Further, assume that  $(i, k) \neq (j, j)$ . Then  $e_{i,k} \neq e_{j,j}$  and so  $[e_{i,j}, e_{j,k}] \neq 0$ .

By definition we get  $e_{i,k} \in L(D, \alpha)$ , and so  $(i, k) \in E$ , i.e., (7) is satisfied.

Letting  $c = \alpha((i, k))$ , we see that  $e_{i,j} \in L_a$ ,  $e_{j,k} \in L_b$ , and  $e_{i,k} \in L_c$ . Since  $L(D, \alpha) = L_0 + L_1$  is a superalgebra, it follows that  $[e_{i,j}, e_{j,k}] \in [L_a, L_b] \subseteq L_{a+b}$ . Therefore  $e_{i,k} - (-1)^{ab}e_{j,k}e_{i,j} \in L_{a+b}$ , and so  $e_{i,k} \in L_{a+b}$ . Since  $e_{i,k} \in L_c$ , we get  $a + b = c$ . Thus (8) is satisfied too.

(iii) $\Rightarrow$ (i) Suppose that conditions (7) and (8) are satisfied.

Let us first show that  $L(D, \alpha)$  is closed for the commutator. Take any elements  $x, y \in L(D, \alpha)$ . Let  $x = \sum_{w \in E} x_w e_w$  and  $y = \sum_{w \in E} y_w e_w$ , where  $x_w, y_w \in F$ . In order to prove that  $[x, y] \in L(D, \alpha)$ , it suffices to verify that  $[x_v e_v, y_w e_w] \in L(D, \alpha)$  for all  $v, w \in E$ .

Putting  $a = \alpha(v)$  and  $b = \alpha(w)$  we get

$$[x_v e_v, y_w e_w] = x_v y_w (e_v e_w - (-1)^{ab} e_w e_v) \tag{9}$$

by (6). Let  $v = (i, j)$  and  $w = (k, \ell)$ . If  $e_v e_w \neq 0$ , then  $j = k$  and (7) implies  $(i, \ell) \in E$ ; whence  $e_v e_w = e_{i,\ell} \in L(D, \alpha)$ . Similarly, if  $e_w e_v \neq 0$ , then  $\ell = i$  and (7) yields  $(k, j) \in E$ ; therefore  $e_w e_v = e_{k,j} \in L(D, \alpha)$ . Thus  $[x_v e_v, y_w e_w] \in L(D, \alpha)$ , and so  $L(D, \alpha)$  is closed under the commutator.

Let  $L_0$  and  $L_1$  be the spaces introduced in the definition of  $L(D, \alpha)$ . Condition (8)

implies that  $L(D, \alpha) = L_0 + L_1$  is a superalgebra. Hence it follows from [1], §1.6, that  $L(D, \alpha)$  satisfies all axioms of a Lie superalgebra.

The implications (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (ii) are similar to the implications (i) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) above, and we omit their proof.  $\square$

A *directed cycle* in  $D = (V, E)$  is a sequence of vertices  $v_1, v_2, \dots, v_m$  such that  $(v_1, v_2), (v_2, v_3), \dots, (v_{m-1}, v_m), (v_m, v_1)$  are edges. A graph is *acyclic* if it has no directed cycles. We are going to reduce the problem of finding a superbasis to the case of acyclic graphs. The following notation is needed for that.

Let  $D = (V, E)$  be a graph with transitive set  $E$  of edges. A *clique* of  $D$  is a complete subgraph of  $D$ , that is a subgraph with all (directed) edges including loops. Denote all maximal cliques of  $D$  by  $M_1, \dots, M_m$ . Choose vertices  $u_1, \dots, u_m$  so that  $u_1 \in M_1, \dots, u_m \in M_m$ . For  $i = 1, \dots, m$ , let

$$C_i = \{u_i\} \times (M_i \setminus \{u_i\})$$

be the set of all edges  $(u_i, w)$ , where  $w$  runs over  $M_i \setminus \{u_i\}$ . Put

$$B_C = C_1 \cup C_2 \cup \dots \cup C_m.$$

Denote by  $A = (V_A, E_A)$  the subgraph induced in  $D$  by the set  $V_A = \{u_1, \dots, u_m\}$  with all loops deleted.

The transitivity of  $E$  implies that  $E_A$  is transitive, too. If  $A$  has a cycle, then (7) shows that all vertices of the cycle belong to one clique of  $D$ . The maximality of the cliques  $M_i$  implies that the cycle is a loop. Since all loops have been removed from  $A$ , it follows that  $A$  is acyclic.

**Proposition 1** *The set  $B_A$  is a superbasis of the acyclic graph  $A$  if and only if  $B = B_A \cup B_C$  is a superbasis of  $D$ .*



*Proof.* For each vertex  $v \in V$ , denote by  $A(v)$  the vertex  $u_i$  chosen in the maximal clique  $M_i$  containing  $v$ .

The ‘if’ part. Suppose that  $B_A \cup B_C$  is a superbasis of  $D$ . Consider any mapping  $\beta_A : B_A \rightarrow \mathbb{Z}_2$ . Define a mapping  $\beta : B \rightarrow \mathbb{Z}_2$  by putting, for each  $b \in B$ ,

$$\beta(b) = \begin{cases} \beta_A(b) & \text{if } b \in B_A, \\ 0 & \text{if } b \in B_C. \end{cases}$$

By Corollary 1 and the definition of a superbasis,  $\beta$  uniquely extends to a homomorphism  $\gamma : E \rightarrow \mathbb{Z}_2$ . Obviously, the restriction  $\gamma_A : E_A \rightarrow \mathbb{Z}_2$  of  $\gamma$  on  $E_A$  is a homomorphism extending  $\beta_A$ . It follows from Corollary 1 that  $B_A$  is a superbasis of  $A$ .

The ‘only if’ part. Suppose that  $B_A$  is a superbasis of  $A$ . Consider any mapping  $\beta : B \rightarrow \mathbb{Z}_2$ . Denote by  $\beta_A : B_A \rightarrow \mathbb{Z}_2$  the restriction of  $\beta$  on  $B_A$ . Corollary 1 implies that  $\beta_A$  uniquely extends to a homomorphism  $\gamma_A : E_A \rightarrow \mathbb{Z}_2$ .

Let us define a mapping  $\gamma : E \rightarrow \mathbb{Z}_2$  by taking any edge  $(u, v) \in E$  and putting

$$\gamma((u, v)) = \beta((A(u), u)) + \gamma_A((A(u), A(v))) + \beta((A(v), v)), \quad (10)$$

where we assume that the images of all loops are defined and are equal to 0 in order to unify several cases and simplify notation. This definition makes sense since each of the edges  $(A(u), u)$  and  $(A(v), v)$  either is a loop or lies in  $B_C$ , and  $(A(u), A(v))$  either is a loop or belongs to  $B_A$ . It is straightforward to verify that  $\gamma$  is a homomorphism from  $E$  to  $\mathbb{Z}_2$  extending the mapping  $\beta$ .

It is also easily seen that the extension is unique. Indeed, for any other homomorphism  $\gamma'$  extending  $\beta$ , the definition of homomorphism shows that  $\gamma'$  is equal to 0 on all loops and satisfies the same equality as (10), and therefore  $\gamma'$  coincides with  $\gamma$  defined above. Hence it follows from Corollary 1 that  $B$  is a superbasis of  $D$ .  $\square$

The linear space spanned by a set  $B$  over  $\mathbb{F}_2$  is denoted by  $\mathbb{F}_2[B]$ . The natural

embedding of  $B$  into  $\mathbb{F}_2[B]$  is denoted by  $\tau = \tau_B$  and is defined by  $\tau(b) = b$  for all  $b \in B$ .

**Lemma 1** *Let  $D = (V, E)$  be an acyclic graph, and let  $B$  be a subset of  $V$  such that the natural embedding  $\tau : B \rightarrow \mathbb{F}_2[B]$  extends to a homomorphism  $\eta : E \rightarrow \mathbb{F}_2[B]$ . Then every mapping  $\beta : B \rightarrow \mathbb{Z}_2$  extends to a homomorphism  $\alpha : E \rightarrow \mathbb{Z}_2$ . If  $\eta$  is unique, then every  $\beta$  extends to a unique  $\alpha$ .*

*Proof.* For each edge  $w \in E$ , the image  $\eta(w) \in \mathbb{F}_2[B]$  can be represented as a sum  $\eta(w) = \sum_{b \in B} w_b b$ , where  $w_b \in \mathbb{F}_2$ . Let us define the mapping  $\alpha$  by the rule

$$\alpha(w) = \sum_{b \in B} w_b \beta(b). \quad (11)$$

It is easily seen that  $\alpha : E \rightarrow \mathbb{Z}_2$  is a homomorphism extending  $\beta$ .

Now, suppose that  $\eta$  is unique. Consider arbitrary homomorphisms  $\alpha', \alpha''$  from  $E$  to  $\mathbb{Z}_2$  extending  $\beta : B \rightarrow \mathbb{Z}_2$ . Let  $\ell = |B|$  and  $B = \{b_1, \dots, b_\ell\}$ . For  $i = 1, \dots, \ell$ , denote by  $\beta_i$  the mapping from  $B$  to  $\mathbb{Z}_2$  such that  $\beta_i(b_i) = 1$  and  $\beta_i(b_1) = \dots = \beta_i(b_{i-1}) = \beta_i(b_{i+1}) = \dots = \beta_i(b_\ell) = 0$ . As we have just verified,  $\beta_i$  extends to a homomorphism  $\alpha_i : E \rightarrow \mathbb{Z}_2$ . Clearly, the sum

$$\mu_i = \alpha_i + \alpha' + \alpha''$$

also is a homomorphism from  $E$  to  $\mathbb{Z}_2$ . Since  $\alpha'$  and  $\alpha''$  coincide with  $\beta$  on  $B$ , we get  $(\alpha' + \alpha'')(b) = 0$  for all  $b \in B$ . Therefore  $\mu_i(b_i) = 1$  and  $\mu_i(b_j) = 0$  for all  $j \neq i$ . It follows that the mapping  $\mu_{\alpha', \alpha''} : E \rightarrow \mathbb{F}_2[B]$  defined for  $w \in E$  by

$$\mu_{\alpha', \alpha''}(w) = \mu_1(w)b_1 + \dots + \mu_\ell(w)b_\ell \in \mathbb{F}_2[B]$$

is a homomorphism extending the natural embedding  $\tau$ . Therefore  $\mu_{\alpha', \alpha''} = \eta$  since  $\eta$  is unique.

Substituting  $\alpha$  for  $\alpha'$  and  $\alpha''$ , we get  $\mu_{\alpha, \alpha} = \eta$ . Similarly,  $\mu_{\alpha, \alpha'} = \eta$ . Hence, for every  $w$  in  $E$  and any  $i$ , we get  $\alpha_i(w) + \alpha(w) + \alpha(w) = \alpha_i(w) + \alpha(w) + \alpha'(w)$ , and so  $\alpha(w) = \alpha'(w)$ .

Thus  $\alpha$  is uniquely defined by  $\beta$ . This completes our proof.  $\square$

A *topological labeling* of the graph  $D = (V, E)$  is an assignment of numbers  $1, 2, \dots, n = |V|$  to vertices so that  $a < b$  for each edge  $(a, b)$ . Every acyclic graph can be topologically labeled, and so we may assume that the original notation for vertices of an acyclic graph is its topological labeling (see [17], §4.5.1).

The *indegree* and *outdegree* of a vertex  $v \in V$  are defined by

$$\text{indeg}(v) = |\{w \in V \mid (w, v) \in E\}|,$$

$$\text{outdeg}(v) = |\{w \in V \mid (v, w) \in E\}|.$$

A vertex of  $D$  is said to be a *source* if  $\text{indeg}(v) = 0$  and  $\text{outdeg}(v) > 0$ .

**Proposition 2** *Let  $D = (V, E)$  be an acyclic graph with transitive set  $E$  of edges. Then  $D$  has a superbasis  $B$  such that each edge of  $B$  begins in a source of  $D$ .*

*Proof.* We may assume that the vertices of  $V$  have been topologically ordered, and that  $V_S = \{1, \dots, \ell\}$  is the set of all sources of  $D$ . For each  $i = 1, \dots, n$ , put  $V(i) = \{1, \dots, i\}$ ,  $E(i) = E \cap (V(i) \times V(i))$  and  $D(i) = (V(i), E(i))$ .

Let us define sets of edges  $B(i)$  by induction on  $i$ . Put  $B(1) = \emptyset$ . Clearly,  $B(1)$  is a superbasis of the null graph  $D(1)$ .

Suppose that a superbasis  $B(i)$  of the graph  $D(i)$  has been found, where all edges of  $B(i)$  begin in sources of  $D$ , and that homomorphism  $\eta_i : E(i) \rightarrow \mathbb{F}_2[B(i)]$  extending the natural embedding  $\tau_i : B(i) \rightarrow \mathbb{F}_2[B(i)]$  has been defined. Consider  $i + 1$  and the subgraph  $D(i + 1)$ . We are going to add into  $B(i)$  some edges leading from sources to  $i + 1$  until the desired  $B(i + 1)$  is obtained.

To this end we proceed by nested induction on the number  $d$  of sources that have been taken care of in the process of adding edges leading to  $i + 1$ . Initially,  $d = 0$  since no sources

have been regarded yet. Here we let  $B(0, i+1) = B(i)$ ,  $T(0, i+1) = E(i)$ , and introduce mappings  $\eta_{0,i+1} = \eta_i$  and  $\tau_{d,i+1} = \tau_i$  to start the process. By the induction assumption for  $i$  we see that  $T(0, i+1)$  is a transitive set of edges and  $\eta_{0,i+1} : T(0, i+1) \rightarrow \mathbb{F}_2[B(0, i+1)]$  is the unique homomorphism extending the natural embedding  $\tau_{0,i+1} : B(0, i+1) \rightarrow \mathbb{F}_2[B(0, i+1)]$ .

Suppose that for some  $0 \leq d < \ell$  the first  $d$  sources from 1 to  $d$  have been viewed and the sets  $B(d, i+1)$  and  $T(d, i+1)$  have been introduced such that  $T(d, i+1)$  is a transitive set of edges and there exists a unique homomorphism  $\eta_{d,i+1} : T(d, i+1) \rightarrow \mathbb{F}_2[B(d, i+1)]$  extending the natural embedding  $\tau_{d,i+1} : B(d, i+1) \rightarrow \mathbb{F}_2[B(d, i+1)]$ . Thus  $B(d, i+1)$  is a superbasis of the graph  $(V(i), T(d, i+1))$  by Lemma 1.

Consider the next source  $d+1$ , and put  $\text{Out}(d+1) = \{v \mid (d+1, v) \in E\}$ . The following three cases are possible.

Case 1. If  $(d+1, i+1) \notin E$ , then we put  $B(d+1, i+1) = B(d, i+1)$ ,  $T(d+1, i+1) = T(d, i+1)$  and  $\eta_{d+1,i+1} = \eta_{d,i+1}$ .

Case 2.  $(d+1, i+1) \in E$  and if there exists  $w \in \text{Out}(d+1) \cap V(i)$  such that  $(w, i+1) \in T(d, i+1)$ . Then we put  $B(d+1, i+1) = B(d, i+1)$ ,

$$T(d+1, i+1) = T(d, i+1) \cup \{(d+1, i+1)\},$$

and define

$$\eta_{d+1,i+1}(x) = \begin{cases} \eta_{d,i+1}(x) & \text{if } x \in T_{d,i+1}, \\ \eta_{d,i+1}((d+1, w)) + \eta_{d,i+1}((w, i+1)) & \text{if } x = (d+1, i+1). \end{cases}$$

Case 3.  $(d+1, i+1) \in E$  and there do not exist any  $w \in \text{Out}(d+1) \cap V(i)$  such that  $(w, i+1) \in T(d, i+1)$ . Then we put  $B(d+1, i+1) = B(d, i+1) \cup (d+1, i+1)$ ,  $T(d+1, i+1) = T(d, i+1) \cup \{(d+1, i+1)\}$ , and define

$$\eta_{d+1,i+1}(x) = \begin{cases} \eta_{d,i+1}(x) & \text{if } x \in T_{d,i+1}, \\ x & \text{if } x = (d+1, i+1). \end{cases}$$

In addition in Cases 2 and 3 we also add to  $T(d+1, i+1)$  all edges  $(z, i+1)$  such that  $z \in \text{Out}(d+1) \cap V(i)$  and  $(z, i+1) \in E$  extending the map  $\eta_{d+1, i+1}$  on these additional edges by putting

$$\eta_{d+1, i+1}(z, i+1) = \eta_{d+1, i+1}((d+1, z)) + \eta_{d, i+1}((z, i+1)).$$

This is consistent with the definitions already given above and takes care of all edges in  $E(i)$ .

In all three cases with the induction assumption it is straightforward that  $T(d+1, i+1)$  is a transitive set of edges and that  $\eta_{d+1, i+1} : T(d, i+1) \rightarrow \mathbb{F}_2[B(d, i+1)]$  is the unique homomorphism extending the natural embedding  $\tau_{d+1, i+1} : B(d, i+1) \rightarrow \mathbb{F}_2[B(d, i+1)]$ . Hence Lemma 1 implies that  $B(d+1, i+1)$  is a superbasis of the graph  $(V(i), T(d+1, i+1))$  too.

Further, since  $T(\ell, i+1) = E(i)$ , it follows that  $B(i+1) = B(\ell, i+1)$  is a superbasis of  $D(i+1)$ . Finally, induction on  $i$  shows that  $B = B(n)$  is a superbasis of  $D = D(n)$ . This completes our proof.  $\square$

*Proof of Theorem 2 follows from Propositions 1 and 2.  $\square$*

## 4 Examples

The examples of full matrix Lie superalgebras and upper triangular matrix Lie superalgebras show that graphs may have multiple superbases, and the same set of edges can be a superbasis for several graphs containing it.

**Example 1** Let  $K_n = (V_n, E_n)$  be the complete graph with  $V_n = \{1, \dots, n\}$ , where  $E_n$

contains all edges including loops. It follows from the proof of Proposition 1 that the set

$$\{(1, i) \mid i = 2, \dots, n\} \quad (12)$$

is a superbasis of  $K_n$ . The proof of a simple claim in Section 1 of [6] shows that the set

$$\{(1, 2), (2, 3), \dots, (n-1, n)\} \quad (13)$$

is a superbasis of  $K_n$  too. This gives a complete description of all block matrix Lie superalgebras defined on the set  $M_n(F)$  of all  $(n \times n)$  matrices over  $F$ . Please note that the essence of the notion of a superbasis has been ubiquitous in various considerations related to matrix superalgebras. Originally the author used the set (13) due to [6], but during a seminar talk at the University of Wisconsin Professor Don Passman instantly suggested the set (12) with easy proof as part of folklore knowledge.

**Example 2** Let  $T_n = (V_n, E_n)$  be the topologically ordered tournament, i.e., the graph with  $V_n = \{1, \dots, n\}$  and

$$E_n = \{(i, j) \mid 1 \leq i \leq j \leq n\}.$$

Now the proof of Proposition 2 shows that the set (12) is a superbasis of  $T_n$  too. Similarly, the proof of the main theorem of [6] also implies that the set (13) is a superbasis of  $T_n$ . This provides a description of all block matrix Lie superalgebras defined on the set  $U_n(F)$  of all upper triangular  $(n \times n)$  matrices over  $F$ .

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